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RING EXPANSIONS ON INTERMEDIATE RINGS UNDER DIFFERENT SITUATIONS

Dr. P. R. PARIHAR

Associate Professor

S.P.C. Government College, Ajmer

Abstract

It is assumed that all rings are commutative with nonzero unity, and that all ring extensions are unital with the same unity. In 1970, Ferrand and Olivier laid the groundwork for what would become known as the study of the intermediate rings of a ring extension. They defined the idea of minimum ring extension as follows: a ring extension R T is said to be minimal if there is no valid ring between R and T. In other words, a minimal ring extension is a ring extension that has the smallest possible size. In later work, Dobbs provided proof that every ring has a ring extension T such that R T is a minimum ring extension. This was done by proving that every ring R has a ring extension T. Azarang et al. designated this subring R by maximum subring. The reasoning for this new term may be understood by considering the fact that not every ring possesses a maximum subring. Take Z as an example. Azarang and colleagues looked into the rings to see whether they had any maximum subrings. When algebraists started working on maximum non-P-subrings of a domain with well-known ring theoretic characteristics P in the literature, such as P=: PID, Pr ufer, valuation, pseudovaluation, Jaffard, ACCP, etc., the study of intermediate rings became more interesting. This is because the study of intermediate rings gained greater attention. They investigated a number of ring theoretic characteristics, some of which cannot be met by a suitable subring R of a domain T but can be satisfied by any intermediate domain, including T.

keywords: *Ring ,expansions ,intermediate*

Introduction

Commutative rings are a subfield of abstract algebra that are concerned with the process of multiplication. It is the mechanism that allows addition and multiplication to conduct themselves in an acceptable manner. Commutative algebraists have demonstrated a significant amount of interest in the study of commutative ring extensions with the same nonzero identity over the course of the past two decades. Research on the ring-theoretic qualities of a ring extension's intermediate rings is going to be carried out. Numerous well-known ring theoretic features, such as valuation, Prufer, and, amongst others, have been investigated on intermediate rings. On ring extensions, several topological characteristics were investigated, as well.

Maximal subrings of a ring

In this study, we focus on maximum subrings of a ring and their properties. Let us assume that R is a valid subring of the ring T. If there is no suitable intermediary ring between R and T, denoted by the notation "[R, T] = 2," one of the two statements is true: either R is considered to be a maximum subring of T, or T is considered to be a minimum ring extension of R. Through the classification of minimum ring extensions of a von Neumann regular ring, we were able to solve an open problem brought up by Dobbs in. In addition to this, we demonstrate that the maximal subrings may be found in the ring R(+)R for every ring R. Additionally, we come up with maximum subrings of R R. We challenge several previously established findings on limited ring extensions. We provide the concept of pointwise maximum subrings and investigate the features of this idea. We demonstrate fundamental characteristics and characteristics of pointwise maximum subrings of rings by our proofs. The pullback features of pointwise maximum subrings of a ring are investigated in this paper. The number of overrings and the Krull dimension of a ring's pointwise maximum subrings are topics that we cover when certain requirements are met. In this lesson, we will go over the pointwise maximum subrings that a polynomial ring can have. The integrally closed pointwise maximum subrings of a ring are characterized here by our team. In conclusion, we show that the G-invariant property of pointwise maximum subring can be proven. The information presented in this chapter was subsequently reported in.

2. Finiteness Conditions on [R, S]

In order to establish a connection between the different finiteness criteria that are imposed on the set [R, S] of intermediate rings, we need to take into consideration the set [R, S] ordered by inclusion. Let P indicate a generic ordered set. When every two different items inside a subset C of P are similar to one another, we refer to that subset as a chain. When any two different members from a subset A of P cannot be compared to one another, we say that the subset A is an antichain. If there is such a thing as an antichain in P, then the size of that antichain is referred to as the width w(P) of P. If there isn't such an antichain in P, then the width is assumed to be (see [12] page 36). The Dilworth's Chain Decomposition Theorem provides a useful relationship between chains and antichains in ordered sets by stating the following: See [12], page 49, for more information on how every ordered set P of width k may be represented as the union of k chains C1,..., Ck P. We refer to the ring R[x] as a straightforward extension of R in S for every x that falls inside the range of SR. Two straightforward extensions, denoted R[x] and R[y], are regarded as being distinct from one another if and only if R[x] 6= R[y]. Recall that a ring extension R S is said to be (strongly) affine if every intermediate ring can be finitely constructed as a ring over R. This is one of the requirements for the term to apply. In order to formulate our outcomes, we need to first provide the definition that is as follows:

Definition 2.1. If there is a finite subset, then we say that the ring extension R S is uniformly affine, sometimes abbreviated as UA. G = {g1, g2, ..., gu} of elements of S such that for every T ∈ [R, S] there is a subset H of G so that T = R[H].

Now that we have reached this point, we are able to provide our first result on ring extensions using just a finitely large number of intermediary rings.

Theorem 2.2. Let $R \subseteq S$ be a ring extension. Then the following statements are equivalent.

(1) [R, S] is finite.

(2) The set $\{R[x] : x \in S\backslash R\}$ of all distinct simple extensions of R in S is finite.

(3) The extension $R \subseteq S$ is a UA extension.

(4) $R \subseteq S$ is an FC extension and w[R, S] is finite.

Proof. (1) ⇒ (2) is trivial.

(2) ⇒ (3): Let F = {s1, s2, ..., sn} be a subset of S so that {R[s1], R[s2], ..., R[sn]} is the set of simple extensions of R in S, and let T ∈ [R, S]. Assume, after a suitable reordering of the elements of F, that s1, s2, ..., sm are the elements of F which are in T. We necessarily have T = R[s1, s2, ..., sm]. Indeed if u ∈ T\R[s1, s2, ..., sm], then R[u] ∈ { / R[s1], R[s2], ..., R[sm]}. Also R[u] ∈ { / R[sm+1], R[sm+2], ..., R[sn]} as otherwise si ∈ T for some i ∈ {m + 1, ..., n} which contradicts the choice of s1, s2, ..., sm in F. Therefore R ⊆ S is a UA extension

(3) ⇒ (4): Since every intermediate ring is generated over R by a subset of a finite set G = {g1, g2, ..., gu} of S, it is easy to see that the set of intermediate rings is finite and therefore every chain and every antichain of [R, S] is finite. That is R ⊆ S is a FC extension and w[R, S] is finite.

(4) ⇒ (1). Let w = w[R, S] be the width of the set of intermediate rings [R, S] ordered by inclusion and let C1, C1, ..., Cw the chains from [R, S] into which [R, S] is decomposed. Then R ⊆ S is a FO extension as |[R, S]| ≤ Σ w i=1|Ci |.

In the case of the ring extension R ⊆ K, where K is the field of quotients of R, we obtain the following result.

Corollary 2.3. Let R be an integral domain, and let K be the field of quotients of R. Then the following statements are equivalent.

(1) [R, K] is finite.

(2) The set {R[x] : x ∈ K\R} of all distinct simple extensions of R in K is finite.

(3) The extension R ⊆ K is a UA extension.

(4) R is an FC domain and w[R, K] is finite.

An ordered set P is said to fulfill the ascending (respectively descending) chain condition, abbreviated as a.c.c. (respectively d.c.c.), if and only if every ascending (respectively descending) chain of elements in P remains stable. This is something that we have learned in the past. It is claimed that the maximal (respectively, the minimal) condition is satisfied by an ordered set P if and only if every nonempty subset of P contains a maximal (respectively, a minimal) member. It is common knowledge that the a.c.c. condition of P is satisfied if and only if it satisfies the maximum condition; and that the d.c.c. condition of P is satisfied if and only if it satisfies the minimum condition. Both of these statements are true.

Remark 2.4. It is not difficult to understand that a ring extension R S is an FC extension iff and only if a.c.c. and d.c.c. with regard to the ordinary set inclusion are met in [R, S], and iff and only if every nonempty collection of intermediate rings in [R, S] includes both a maximum element and a minimum element.

The Integrally Closed Case

In this part of the article, we are going to use the assumption that R is integrally closed in S, and for our characterizations, we are going to employ residually algebraic pairings. Keep in mind that a ring extension is a R ⊆ T is called a residually algebraic extension, [5], if for each prime ideal Q of T, T /Q is algebraic over R/(Q ∩ R). We say that (R, S) is a residually algebraic pair, Definition 2.1 of [1], if for each T ∈ [R, S], R ⊆ T is a residually algebraic extension. We say that (R, S) is a normal pair, [4], if for each T ∈ [R, S], T is integrally closed in S. The extension R ⊆ S is called a primitive extension (or P-extension, see [8]) if each element u of S is a root of a polynomial f(X) ∈ R[X] with unit content; i.e. the coefficients of f(X) generate the unit ideal of R. Residually algebraic pairs (R, S) with R integrally closed in S are necessarily normal pairs by Theorem 2.5(vi) of [1], therefore they enjoy several nice properties. The following properties of normal pairs are going to be used in this paper.

Remark 3.1. Let (R, S) be a normal pair, let T ∈ [R, S] be any intermediate ring, and let M ax(R) = {Mi : i ∈ I} be the set of maximal ideals of R. Then the following properties hold true.

(1) *(Proposition 4 of [4], and Lemma 3.1 of [1]) The intermediate ring T is the intersection of some localizations of R, more precisely for each maximal R-ideal M_i, there is a prime R-ideal Q_i such that $T_{R\setminus M_i} = R_{Q_i}$ and $T = \cap_{i\in I} T_{R\setminus M_i} = \cap_{i\in I} R_{Q_i}$.*
(2) *(Lemma 3.1 of [1]) $Spec(T) = \{PT : P \in Spec(R) \text{ and } P \subseteq Q_i \text{ for some } i \in I\}$.*
(3) *(Lemma 3.1 of [1]) $Max(T)$ is the subset of maximal elements in the set $\{Q_iT : i \in I\}$.*
(4) *([4]) Each prime ideal of T is an extension PT of a prime ideal P of R. Furthermore, for each prime Q of R such that $QT \neq T$, QT is a prime ideal of T satisfying $QT \cap R = Q$ and $T_{QT} = R_{QT\cap R} = R_Q$.*

For a ring extension R ⊆ S, we denote by Φ : Spec(S) → Spec(R) the canonical contraction map, i.e. Φ(P) = P ∩ R, for every P in Spec(S). It is easy to see that FO extensions are necessarily FC extensions. The following conclusion demonstrates that both ideas are identical to one another as well as to a number of other characterizations when it comes to integrally closed extensions of integral domains.

Theorem 3.2. Let R ⊆ S be an extension of integral domains, and let A = A(R, S) := {P ∈ Spec(R) : P 6⊂ N ∩ R, ∀ N ∈ M ax(S)}. If R is integrally closed in S, then the following statements are equivalent.

(1) $[R, S]$ *is finite.*
(2) *The set* $\{R[x] : x \in S \setminus R\}$ *of all simple extensions of* R *in* S *is finite.*
(3) *The extension* $R \subseteq S$ *is a UA extension.*
(4) $R \subseteq S$ *is an FC extension and* $w[R, S]$ *is finite.*
(5) $R \subseteq S$ *is an FC extension.*
(6) (R, S) *is a residually algebraic pair such that* $A \setminus \Phi(Max(S))$ *is finite.*
(7) (R, S) *is a residually algebraic pair such that* $Max(R) \setminus \Phi(Max(S))$ *and* $dim A$ *are finite.*
(8) *a.c.c. and d.c.c. are satisfied in* $[R, S]$.
(9) *Every non-empty collection of intermediate rings in* $[R, S]$ *has both a maximal element and a minimal element.*

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3)$\Leftrightarrow$ (4) is satisfied by Theorem 2.2. Also (4) $\Rightarrow$ (5) and (5) $\Leftrightarrow$ (8)$\Leftrightarrow$ (9) are trivial. It is enough to show that (5) $\Rightarrow$ (7) $\Leftrightarrow$ (6) $\Rightarrow$(1).

(5) $\Rightarrow$ (7): Let $Max(R) = \{M_i : i \in I\}$ be the set of maximal ideals of R. We first note that if (R, S) is a normal pair, then

$A(R, S) = \{P \in Spec(R) : P \not\subset Q_i,\ \forall\, i \in I\}$,

where Q_i is the prime of R defined in Remark 3.1 for which $S_{R \setminus M_i} = R_{Q_i}$.

Indeed by Remark 3.1 $Max(S)$ is the subset of maximal elements in the set $\{Q_i S : i \in I\}$, hence let J be the subset of I such that $Max(S) = \{Q_i S : i \in J\}$. For $P \in Spec(R)$, we have

$P \not\subset Q_i,\ \forall\, i \in I \Leftrightarrow$
$P \not\subset Q_i,\ \forall\, i \in J \Leftrightarrow$
$P \not\subset Q_i S \cap R,\ \forall\, i \in J \Leftrightarrow$
$P \not\subset N \cap R,\ \forall\, N \in Max(S) \Leftrightarrow$
$P \in A(R, S)$.

Therefore {P ∈ Spec(R) : P 6⊂ Qi , ∀ i ∈ I} = A(R, S).

Now if R ⊆ S is an FC extension, then [R, S] satisfy d.c.c. and R ⊆ S is a primitive extension by [7], Proposition 1.1. Thus (R, S) is a residually algebraic pair by [6], Theorem 6.5.7; that is (R, S) is a normal pair as R is integrally closed in S by [1], Theorem 2.5. Finally (R, S) is a normal pair such that M ax(R) \ Φ(M ax(S)) and dim A are finite by [10], Theorem 2.1.

(7) ⇔ (6) ⇒ (1) by Theorem 2.1 of [10] as in this case (R, S) is a normal pair and A = {P ∈ Spec(R) : P 6⊂ Qi , ∀ i ∈ I} as in the proof of (5)⇒(7).

From the equivalence of (5), (6) and (1) in Theorem 3.2, we obtain the following:

Corollary 3.3. Let R ⊆ S be an FC ring extension, such that R is integrally closed in S. Then (R, S) is a normal pair with only finitely many intermediate rings.

From the equivalence of (4) and (5) in Theorem 3.2, we obtain the following:

Corollary 3.4. Let R ⊆ S be a ring extension, such that R is integrally closed in S. If every chain of intermediate rings is finite, then also every antichain of intermediate rings is finite.

Remark 3.5. The converse of Corollary 3.4 is not true in general as it is enough to consider the extension R ⊆ K, where R is a valuation domain of infinite dimension. This extension has w[R, S] = 1, but is not FC.

Remark 3.6. If (R, S) is assumed to be a normal pair, then A = {P ∈ Spec(R) : P 6⊂ Qi , ∀ i ∈ I} as in the proof of Theorem 3.2. The statements (1), (5), (6), and (7) of Theorem 3.2 are the same as Theorem 2.1 of [10].

In the case of the ring extension R ⊆ K, where K is the field of quotients of R, we have SR\Mi = KR\Mi = R{0} and the ideals Qi defined by Remark 3.1−1 are all equal to the {0}-ideal. Hence A = {P ∈ Spec(R) : P 6⊂ {0}, ∀ i ∈ I} = Spec(R)\{0}. Thus we obtain the following result.

Corollary 3.7. Let R be an integrally closed domain, and let K be the field of quotients of R. Then the following are equivalent:

(1) R is an FO domain.
(2) The set {R[s] : s ∈ K} of all distinct simple extensions of R in its quotient field is finite.
(3) The extension R ⊆ K is a UA extension.
(4) R is an FC domain and w[R, K] is finite.
(5) R is an FC domain.
(6) R is a Pr¨ufer domain with finite spectrum.
(7) R is a Pr¨ufer domain such that M ax(R) and dimR are finite.
(8) a.c.c. and d.c.c. are satisfied in [R, K].
(9) Every non-empty collection of overrings of R has both a maximal element and a minimal element.

Remark 3.8. The statements (1), (5), (6), and (7) of Corollary 3.7 are the same as Theorem 1.5 of [7], which in turn was a generalization of Corollary 2.1 of [10].

Theorem 3.9. Let R ⊆ S be an extension of integral domains, and let R0 be the integral closure of this extension. R in S. If R ⊆ S is an FC extension then the following statements hold true.

(1) (R, S) is a residually algebraic.

(2) (R0 , S) is a normal pair.

(3) [R0 , S] is finite; i.e. R0 ⊆ S is an FO extension.

Proof. (1) Denote by the symbol R0 the integral closure of the region R contained within the ring S. Then each chain of intermediate rings in [R0, S] is also finite, and as a result, according to Theorem 3.2, the pair [R0, S] is a residually algebraic pair. This is similar to the statement that (R, S) is a residually algebraic pair according to Remark 2.2 of [1]. (2) The pair R0, S is both a residually algebraic pair and a normal pair, in the same way that every other residually algebraic pair, R, S, in which R is integrally closed in S is also a normal pair. Because it is also an FC extension, the expression (3) (R0, S) qualifies as a FO extension according to Theorem 3.2.

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